9. Define Poisson Distribution.

Solution : A random variable X is said to follow a Poisson Distribution if it assumes only non-negative values and its probability mass function is given by :

$$p(x,\lambda) = P(X = x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}; & x = 0,1,2,\ldots;\ \lambda > 0 \\ 0, & otherwise \end{cases}$$

where, λ is the parameter of the distribution.

10. What are the limiting cases for which Poisson Distribution is under Binomial Distribution ?

Solution : The required conditions are :

(i) n, the number of trials is indefinitely large, i.e., $n \to \infty$.

(ii) p, the constant probability of success for each trial is indefinitely small, i.e., $p \to 0$.

(iii) $np = \lambda$, where λ is finite.

$$\Rightarrow p = \frac{\lambda}{n}$$

$$\Rightarrow q = 1 - \frac{\lambda}{n}$$, in this case λ is positive real number.

11. Find the probability function of the Poisson Distribution.

Solution : We know that the probability of x successes in a series of n independent trials is :

$$b(x;n,p) = \binom{n}{x} p^x q^{n-x};\quad x = 0,1,2,\ldots,n$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-x+1)}{x!} \cdot \frac{\left(\frac{\lambda}{n}\right)^x}{\left(1-\frac{\lambda}{n}\right)^x} \cdot \left(1 - \frac{\lambda}{n}\right)^n$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{x-1}{n}\right)}{x!\left(1-\frac{\lambda}{n}\right)^x} . \lambda^x \left(1-\frac{\lambda}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} b(x;n,p) = \frac{e^{-\lambda}\lambda^x}{x!};\ \ x = 0,1,2,\ldots;\ \lambda > 0.$$

which is the probability function of the Poisson Distribution.

12. Derive the Moments of Poisson Distribution.

Solution : Here, we have,

$$u_1{}' = \mathrm{E}(X) = \sum_{x=0}^{\infty} x\, p(x,\lambda) = \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!} = \lambda e^{-\lambda}\left\{\sum_{x=0}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}\right\}$$

$$= \lambda e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$

$$= \lambda e^{-\lambda}.e^{\lambda}$$

$= \lambda$ (mean of the Poisson Distribution)

$$u_2{}' = \mathrm{E}(X^2) = \sum_{x=0}^{\infty} x^2\, p(x,\lambda) = \sum_{x=0}^{\infty}\{x(x-1)+x\}.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda}\left\{\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}\right\} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$u_3{}' = \mathrm{E}(X^3) = \sum_{x=0}^{\infty} x^3\, p(x,\lambda) = \sum_{x=0}^{\infty}\{x(x-1)(x-2)+3x(x-1)+x\}.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} + 3\sum_{x=0}^{\infty} x(x-1).\frac{e^{-\lambda}\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^3 e^{-\lambda}\left\{\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!}\right\} + 3\lambda^2 e^{-\lambda}\left\{\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}\right\} + \lambda$$

$$= \lambda^3 e^{-\lambda} e^{\lambda} + 3\lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$u_4{}' = \mathrm{E}(X^4) = \sum_{x=0}^{\infty} x^4\, p(x,\lambda)$$

$$= \sum_{x=0}^{\infty}\{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) +$$

$$7x(x-1) + x\}.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^4 e^{-\lambda}\left\{\sum_{x=4}^{\infty}\frac{\lambda^{x-4}}{(x-4)!}\right\} + 6\lambda^3 e^{-\lambda}\left\{\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!}\right\} + 7\lambda^2 e^{-\lambda}\left\{\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}\right\}$$

$$+ \lambda$$

$$= \lambda^4 e^{-\lambda} e^{\lambda} + 6\lambda^3 e^{-\lambda} e^{\lambda} + 7\lambda^2 e^{-\lambda} e^{\lambda} + \lambda$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are obtained as following ways :

$$\mu_2 = \mu_2' - {\mu_1'}^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Thus, both the mean and the variance of the Poisson Distribution are each equal to λ.

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4$$

$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda(\lambda^2 + \lambda) - 3\lambda^4$$

$$= 3\lambda^2 + \lambda$$

Co-efficients of skewness and kurtosis are given by :

$$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \text{ and } \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2} = 3 + \frac{1}{\lambda} \text{ and } \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$