

## 2.1 Newton-Gregory Formula for Backward Interpolation with Equal Intervals :

Suppose,  $y = f(x)$  is a given function.

Suppose,  $f(a), f(a + h), f(a + 2h), \dots, f(a + nh)$  are  $(n + 1)$  corresponding values for  $(n + 1)$  equidistant values  $a, a + h, a + 2h, \dots, a + nh$  of the independent variable  $x$  respectively, the interval of differencing is  $h$ .

Now, we consider,  $f(x)$  is a  $n$ th degree polynomial in  $x$ , such that

$$\begin{aligned} f(x) &= A_0 + A_1(x - a - nh) + A_2(x - a - nh)(x - a - \overline{n-1}h) \\ &\quad + A_3(x - a - nh)(x - a - \overline{n-1}h)(x - a - \overline{n-2}h) \\ &\quad + \dots + A_n(x - a - nh)(x - a - \overline{n-1}h)(x - a - \overline{n-2}h) \\ &\quad \dots (x - a - h) \end{aligned} \quad \dots \dots \dots \quad (1)$$

where  $A_0, A_1, A_2, \dots, A_n$  constants to be determined.

Putting successively  $x = a + nh, a + (n - 1)h, a + (n - 2)h, \dots, a$  in (1), we have,

$$f(a + nh) = A_0 \quad (\text{when } x = a + nh) \quad \dots \dots \dots \quad (2)$$

$$\therefore A_0 = f(a + nh)$$

$$f(a + \overline{n-1}h) = A_0 + A_1(-h) \quad (\text{when } x = a + \overline{n-1}h)$$

$$\Rightarrow A_1h = A_0 - f(a + \overline{n-1}h)$$

$$\Rightarrow A_1 = \frac{f(a+nh) - f(a+\overline{n-1}h)}{h}$$

$$\therefore A_1 = \frac{\nabla f(a+nh)}{h} \quad \dots \dots \dots \quad (3)$$

$$\begin{aligned}
 f(a + \overline{n-2} h) &= A_0 + A_1(-2h) + A_2(-2h)(-h) \\
 \Rightarrow f(a + \overline{n-2} h) &= A_0 - 2h A_1 + 2h^2 A_2 \quad (\text{when } x = a + \overline{n-2} h) \\
 \Rightarrow 2h^2 A_2 &= f(a + \overline{n-2} h) + 2h \cdot A_1 - A_0 \\
 \Rightarrow 2!h^2 A_2 &= f(a + \overline{n-2} h) + 2h \cdot \frac{\nabla f(a+nh)}{h} - f(a + nh) \\
 \Rightarrow 2!h^2 A_2 &= f(a + nh) - 2f(a + \overline{n-1} h) + f(a + \overline{n-2} h) \\
 \Rightarrow 2!h^2 A_2 &= \nabla^2 f(a + nh) \\
 \therefore A_2 &= \frac{\nabla^2 f(a+nh)}{2!h^2} \quad \dots\dots\dots (4)
 \end{aligned}$$

Similarly, we get

$$A_3 = \frac{\nabla^3 f(a+nh)}{3! h^3}$$

$$A_4 = \frac{\nabla^4 f(a+nh)}{4! h^4}$$

.....

.....

$$A_n = \frac{\nabla^n f(a+nh)}{n! h^n}$$

Now, substituting these values of  $A_0, A_1, A_2, \dots, A_n$  in (1), we get

$$\begin{aligned}
 f(x) &= f(a + nh) + (x - a - nh) \frac{\nabla f(a + nh)}{h} \\
 &\quad + (x - a - nh)(x - a - \overline{n-1} h) \cdot \frac{\nabla^2 f(a + nh)}{2! h^2} \\
 &\quad + (x - a - nh)(x - a - \overline{n-1} h)(x - a - \overline{n-2} h) \frac{\nabla^3 f(a + nh)}{3! h^3} \\
 &\quad + \dots + (x - a - nh)(x - a - \overline{n-1} h)(x - a - \overline{n-2} h) \\
 &\quad \dots \dots \dots (x - a - h) \frac{\nabla^n f(a + nh)}{n! h^n} \quad \dots \dots \dots (5)
 \end{aligned}$$

Then, we put

$$\begin{aligned}x &= a + nh + uh \\ \therefore x - a - nh &= uh \\ x - a - (n - 1)h &= uh + h = (u + 1)h \\ x - a - (n - 2)h &= uh + 2h = (u + 2)h \\ &\dots \\ &\dots \\ x - a - h &= uh + nh - h = (u + n - 1)h\end{aligned}$$

Then, the equation (5) becomes

$$\begin{aligned}
f(x) &= f(a + nh + uh) \\
&= f(a + nh) + uh \frac{\nabla f(a + nh)}{h} \\
&\quad + uh(u + 1)h \cdot \frac{\nabla^2 f(a + nh)}{2! h^2} \\
&\quad + uh(u + 1)h(u + 2)h \cdot \frac{\nabla^3 f(a + nh)}{3! h^3} \\
&\quad + \dots + uh(u + 1)h(u + 2)h \dots (u + n - 1)h \frac{\nabla^n f(a + nh)}{n! h^n} \\
\\
\therefore f(a + nh + uh) &= f(a + nh) + u \nabla f(a + nh) \\
&\quad + \frac{u(u+1)}{2!} \cdot \nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!} \cdot \nabla^3 f(a + nh) \\
&\quad + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!} \cdot \nabla^n f(a + nh) \quad \dots \quad (6)
\end{aligned}$$

This formula is known as Newton-Gregory Formula for Backward Interpolation with equal interval.

## **2.2 Worked out Examples :**

**Example (1) :** From the following statistical data, construct a difference table and from it estimate  $y$ , when  $x = 0.7$

$x :$	0	0.1	0.2	0.3	0.4
$y :$	1	1.095	1.179	1.251	1.310

**Solution :** We consider the given function as  $y = f(x)$ .

From the given data, the difference table will be

$x$	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	1				
0.1	1.095	0.095			
0.2	1.179	0.084	-0.011		
0.3	1.251	0.072	-0.012	-0.001	
0.4	1.310	0.059	-0.013	-0.001	0

Now applying Newton-Gregory Backward Interpolation Formula (Upto third differences, fourth difference is zero), we get

$$f(a + nh + uh) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!} \cdot \nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!} \cdot \nabla^3 f(a + nh) \dots \dots \dots (1)$$

$$\text{Here, } a + nh = 0.4, \quad h = 0.1, \quad x = a + nh + uh$$

$$a + nh + uh = 0.7$$

$$\Rightarrow 0.4 + uh = 0.7$$

$$\Rightarrow 0.1 u = 0.7 - 0.4 = 0.3$$

$$\Rightarrow u = 3$$

From (1), we have

$$\begin{aligned}
 f(a + nh + uh) &= f(0.7) \\
 &= f(0.4) + 3 \times 0.059 + \frac{3(3+1)}{2!} \times (-0.013) \\
 &\quad + \frac{3(3+1)(3+2)}{3!} \times (-0.001) \\
 &= 1.310 + 0.177 - 0.078 - 0.01 \\
 &= 1.399
 \end{aligned}$$

**Example (2) :** Given,  $\log x$  for  $x = 40, 45, 50, 55$  and  $60$  according to the following table :

$x$	:	40	45	50	55	60
$\log x$	:	1.6021	1.6532	1.6990	1.7404	1.7782

Find the value of  $\log 62$ .

**Solution :** We consider the given function as  $y = f(x) = \log x$

From the given data, the difference table will be

$x$	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
40	1.6021				
45	1.6532	0.0511	-0.0053	0.0009	
50	1.6990	0.0458	-0.0044	0.0008	-0.0001
55	1.7404	0.0414	-0.0036		
60	1.7782	0.0378			

Now applying Newton-Gregory Backward Interpolation Formula  
 ( Upto fourth differences ), we get

$$\begin{aligned}
 f(a + nh + uh) &= f(a + nh) + u\nabla f(a + nh) \\
 &\quad + \frac{u(u+1)}{2!} \cdot \nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!} \cdot \nabla^3 f(a + nh) \\
 &\quad + \frac{u(u+1)(u+2)(u+3)}{4!} \cdot \nabla^4 f(a + nh) \quad \dots\dots\dots (1)
 \end{aligned}$$

$$\text{Here, } a + nh = 60, \quad h = 5, \quad x = a + nh + uh$$

$$\begin{aligned}
 a + nh + uh &= 62 \\
 \Rightarrow 60 + uh &= 62 \\
 \Rightarrow 5u &= 2 \\
 \Rightarrow u &= 0.4
 \end{aligned}$$

From (1), we have

$$\begin{aligned}
 f(a + nh + uh) &= f(62) = \log 62. \\
 &= f(60) + 0.4 \times 0.0378 + \frac{0.4(0.4+1)}{2!} \times (-0.0036) \\
 &\quad + \frac{0.4(0.4+1)(0.4+2)}{3!} \times 0.0008 \\
 &\quad + \frac{0.4(0.4+1)(0.4+2)(0.4+3)}{4!} \times (-0.0001) \\
 &= 1.7782 + 0.01512 - 0.00101 + 0.00018 - 0.00002 \\
 &= 1.79247 \\
 &= 1.7925, \text{ approximately}
 \end{aligned}$$

**Example (3) :** The population of a district in Arunachal Pradesh is as follows :

Year	: 1971	1981	1991	2001	2011
Population (in Lakh)	: 20	25	32	45	61

Estimate the increase in population during the period 1995 to 2001.

**Solution :** We consider as

Year implies independent variable  $x$

Population (in Lakh) implies dependent variable  $y$

$$\text{i.e., } y = f(x)$$

From the given data, the difference table will be

Year $x$	Population in Lakh $f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1971	20				
1981	25	5			
1991	32	7	2		
2001	45	13	6	4	
2011	61	16	3	-3	-7

Now applying Newton-Gregory Backward Interpolation Formula  
( Upto fourth differences ), we get

$$f(a + nh + uh) = f(a + nh) + u\nabla f(a + nh) + \frac{u(u+1)}{2!} \cdot \nabla^2 f(a + nh) + \frac{u(u+1)(u+2)}{3!} \cdot \nabla^3 f(a + nh)$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!} \cdot \nabla^4 f(a + nh) \quad \dots \dots \dots (1)$$

Here,  $a + nh = 2011, h = 10, x = a + nh + uh$

$$\begin{aligned} a + nh + uh &= 1995 \\ \Rightarrow 2011 + 10u &= 1995 \\ \Rightarrow 10u &= 1995 - 2011 = -16 \\ \Rightarrow u &= \frac{-16}{10} = -1.6 \end{aligned}$$

From (1), we have

$$\begin{aligned} f(a + nh + uh) &= f(1995) \\ &= f(2011) + (-1.6) \times 16 + \frac{(-1.6)(-1.6+1)}{2!} \times 3 \\ &\quad + \frac{(-1.6)(-1.6+1)(-1.6+2)}{3!} \times (-3) \\ &\quad + \frac{(-1.6)(-1.6+1)(-1.6+2)(-1.6+3)}{4!} \times (-7) \\ &= 61 - 25.6 + 1.44 - 0.192 - 0.1568 \\ \therefore f(1995) &= 36.4912 \end{aligned}$$

The population of the district in 1995 = 36,49,120

The increase in population during the period 1995 to 2001

$$\begin{aligned} &= 45,00,000 - 36,49,120 \\ &= 8,50,880 \end{aligned}$$