Finite Numerical Differences

1.1 Introduction :

Suppose, f is a function from x into y in the domain (a, b) such that $\mathrm{y} = f(x)$. For each value of x in the given interval, the value of y can be calculated. In its broad meaning, the values of the dependent variables are changed due to the changes of independent variables in the given functions.

We consider a given function $\mathrm{y} = f(x)$.
Suppose, we take the values $a, a + h, a + 2h, \ldots\ldots, a + nh$ for independent variable x, with respect to the above function, the values of the dependent variables will be $f(a), f(a + h), f(a + 2h), \ldots\ldots, f(a + nh)$ respectively. More clearly,

x	$f(x)$
a	$f(a)$
$a + h$	$f(a + h)$
$a + 2h$	$f(a + 2h)$
$\vdots$	$\vdots$
$a + nh$	$f(a + nh)$

1.2 Argument and Entry :

We consider a given function $\mathrm{y} = f(x)$.
For every value of the independent variable of the function there exists a corresponding value for the dependent variable.

We consider a, $a+h$, $a+2h$, ……, $a+nh$ etc. are the values for independent variable x, for the above function $\mathrm{y}=f(x)$. Then, their corresponding values will be $f(a)$, $f(a+h)$, $f(a+2h)$, …….. , $f(a+nh)$, etc. respectively for the dependent variable y.

The values of the independent variables are called the Arguments and the values of the corresponding to the arguments are called the Entries.

The difference between two consecutive arguments is called the Interval of Differencing or simply Interval.

1.3 Differences :

We consider a given function $\mathrm{y}=f(x)$.
Suppose, the arguments are taken with an equal interval h.

The first forward difference of $f(x)$ is defined as :

$$\boxed{\Delta\, \boldsymbol{f(x)=f(x+h)-f(x)}}$$

For arguments a, $a+h$, $a+2h$, etc., we shall get,

$$\begin{aligned}\Delta\, f(a) &= f(a+h)-f(a)\\ \Delta\, f(a+h) &= f(a+2h)-f(a+h)\\ \Delta\, f(a+2h) &= f(a+3h)-f(a+2h), \text{ etc.}\end{aligned}$$

The second forward difference of $f(x)$ is defined as :

$$\boxed{\boldsymbol{\Delta^2 f(x)=\Delta\,[\Delta\, f(x)]=\Delta\, f(x+h)-\Delta\, f(x)}}$$

For arguments a, $a+h$, $a+2h$, etc., we shall get,

$$\begin{aligned}\Delta^2 f(a) &= \Delta\,[\Delta f(a)]\\ &= \Delta\,[\,f(a+h)-f(a)]\end{aligned}$$

$$
\begin{aligned}
&= \Delta f(a+h) - \Delta f(a) \\
&= f(a+2h) - f(a+h) - f(a+h) + f(a) \\
&= f(a+2h) - 2f(a+h) + f(a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2 f(a+h) &= \Delta\,[\Delta f(a+h)] \\
&= \Delta\,[\,f(a+2h) - f(a+h)] \\
&= \Delta f(a+2h) - \Delta f(a+h) \\
&= f(a+3h) - f(a+2h) - f(a+2h) + f(a+h) \\
&= f(a+3h) - 2f(a+2h) + f(a+h)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2 f(a+2h) &= \Delta\,[\Delta f(a+2h)] \\
&= \Delta\,[\,f(a+3h) - f(a+2h)] \\
&= \Delta f(a+3h) - \Delta f(a+2h) \\
&= f(a+4h) - f(a+3h) - f(a+3h) + f(a+2h) \\
&= f(a+4h) - 2f(a+3h) + f(a+2h), \text{ etc.}
\end{aligned}
$$

The third forward difference of $f(x)$ is defined as :

$$
\boxed{\boldsymbol{\Delta^3 f(x) = \Delta\,[\Delta^2 f(x)] = \Delta^2 f(x+h) - \Delta^2 f(x)}}
$$

For arguments $a, a+h, a+2h$, etc., we shall get,

$$
\begin{aligned}
\Delta^3 f(a) &= \Delta\,[\Delta^2 f(a)] \\
&= \Delta^2 f(a+h) - \Delta^2 f(a) \\
&= f(a+3h) - 2f(a+2h) + f(a+h) - f(a+2h) \\
&\quad +2f(a+h) - f(a) \\
&= f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^3 f(a+h) &= \Delta\,[\Delta^2 f(a+h)] \\
&= \Delta^2 f(a+2h) - \Delta^2 f(a+h) \\
&= f(a+4h) - 2f(a+3h) + f(a+2h) - f(a+3h) \\
&\quad +2f(a+2h) - f(a+h) \\
&= f(a+4h) - 3f(a+3h) + 3f(a+2h) - f(a+h)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^3 f(a+2h) &= \Delta\,[\Delta^2 f(a+2h)] \\
&= \Delta^2 f(a+3h) - \Delta^2 f(a+2h) \\
&= f(a+5h) - 2f(a+4h) + f(a+3h) - f(a+4h)
\end{aligned}
$$

$$+2f(a+3h) - f(a+2h)$$
$$= f(a+5h) - 3f(a+4h) + 3f(a+3h) - f(a+2h)$$

and so on.

In tabular form these forward differences may be evaluated as given in the Table (A) :

Table (A) : Forward Finite Difference Table

Argument x	Entry $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$
$\boldsymbol{a}$	$\boldsymbol{f(a)}$			
		$\boldsymbol{\Delta f(a)}$		
$\boldsymbol{a+h}$	$\boldsymbol{f(a+h)}$		$\boldsymbol{\Delta^2 f(a)}$	
		$\boldsymbol{\Delta f(a+h)}$		$\boldsymbol{\Delta^3 f(a)}$
$\boldsymbol{a+2h}$	$\boldsymbol{f(a+2h)}$		$\boldsymbol{\Delta^2 f(a+h)}$	
		$\boldsymbol{\Delta f(a+2h)}$		$\boldsymbol{\Delta^3 f(a+h)}$
$\boldsymbol{a+3h}$	$\boldsymbol{f(a+3h)}$		$\boldsymbol{\Delta^2 f(a+2h)}$	
		$\boldsymbol{\Delta f(a+3h)}$		
$\boldsymbol{a+4h}$	$\boldsymbol{f(a+4h)}$			

In the above table, a is the Leading Argument, $f(a)$ is the Leading Entry, $\Delta f(a)$ is the Leading First Difference, $\Delta^2 f(a)$ is the Leading Second Difference, $\Delta^3 f(a)$ is the Leading Third Difference, and so on.

1.4 The Difference Operator Δ :

The symbol Δ is used for forward difference operation. It is an operator. $\Delta^2 f(x)$ stands for second time operations (two times operations) for the function $f(x)$. Similarly, $\Delta^n f(x)$ stands for n times operations for the function $f(x)$.

1.5 Worked out Examples :

Example (1) : If $y = 2x^3 + x^2 - 3x + 1$, calculate the values of y for $x = 0, 1, 2, 3, 4$ and construct the difference table.

Solution :

Here, $y = f(x) = 2x^3 + x^2 - 3x + 1$

When $x = 0$, then $y_0 = f(0) = 1$

When $x = 1$, then $y_1 = f(1) = 1$

When $x = 2$, then $y_2 = f(2) = 15$

When $x = 3$, then $y_3 = f(3) = 55$

When $x = 4$, then $y_4 = f(4) = 133$

Argument x	Entry $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$	4th Difference $\Delta^4 f(x)$
0	1				
		0			
1	1		14		
		14		12	
2	15		26		0
		40		12	
3	55		38		
		78			
4	133				

Example (2) : Construct a forward difference table from the following values of x and y :

x **:**	5	10	15	20	25
y **:**	9960	9835	9700	9595	9520

Solution : Here, $y = f(x)$.

$\therefore$ Forward difference Table is constructed as follows :

Argument x	Entry $f(x)$	1st Difference $\Delta f(x)$	2nd Difference $\Delta^2 f(x)$	3rd Difference $\Delta^3 f(x)$	4th Difference $\Delta^4 f(x)$
5	9960				
		-125			
10	9835		-10		
		-135		40	
15	9700		30		-40
		-105		0	
20	9595		30		
		-75			
25	9520				

Example (3) : Construct a forward difference table, given that

x :	1	2	3	4	5
y :	754	625	510	432	353

Write down the values of $\Delta^2 y_2, \Delta^3 y_3, \Delta^4 y_1$.

Solution : Here, $y = f(x)$.

$\therefore$ Forward difference Table is constructed as follows :

Argument x	Entry y	1st Difference Δy	2nd Difference $\Delta^2 y$	3rd Difference $\Delta^3 y$	4th Difference $\Delta^4 y$
1	754				
		-129			
2	625		14		
		-115		23	
3	510		37		-61
		-78		-38	
4	432		-1		
		-79			
5	353				

In the column of Second Differences, the Second term is $\Delta^2 y_2$ i.e., $\Delta^2 y_2 = 37$.

In the column of Third Differences, the Third term is $\Delta^3 y_3$, but there is no Third term in this column, therefore $\Delta^3 y_3$ does not exists.

In the column of Fourth Differences, the First term is $\Delta^4 y_1$ i.e., $\Delta^4 y_1 = -61$.

Example (4) : Construct the forward difference table and find the 5th term of the series 1, 5, 11, 19.

Solution : Here, $y = f(x)$.

∴ Forward difference Table is constructed as follows :

Argument x	Entry y	1st Difference Δy	2nd Difference $\Delta^2 y$
1	1		
		4	
2	5		2
		6	
3	11		2
		8	
4	19		$\boxed{2}$
		8+2=10	
5	19+10=29		

From the difference table, it is found that the second differences are constant. Therefore, the third term in this column should be considered the same constant value 2 i.e., $\Delta^2 f(3) = 2$. To get this constant value 2, we should add the constant value 2 to the third term in the column of the first differences so that the fourth term in this column is obtained as 8+2=10 i.e.,

$$\Delta f(4) = \Delta f(3) + \Delta^2 f(3) = 8 + 2 = 10$$

Then, $f(5)$ is obtained by adding 10 to $f(4)$ i.e.,

$$f(5) = f(4) + \Delta f(4) = 19 + 10 = 29$$

1.6 Backward Differences :

Suppose, $y = f(x)$ is a given function. We consider, $a, a + h, a + 2h,$ ……, $a + nh$ etc. are the values for independent variable x, then their corresponding values will be $f(a), f(a + h), f(a + 2h),$ …….. , $f(a + nh)$, etc. respectively.

The backward differences of $f(x)$ are defined as :

$$\boxed{\nabla f(x) = f(x) - f(x - h)}$$

Similarly,

$$\nabla^2 f(x) = \nabla f(x) - \nabla f(x - h)$$
$$\nabla^3 f(x) = \nabla^2 f(x) - \nabla^2 f(x - h)$$

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$$\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x - h)$$

In tabular form these backward differences may be evaluated as Given in the Table (B) :

Table (B) : Backward Finite Difference Table

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
a	$f(a)$			
		$\nabla f(a)$		
$a+h$	$f(a+h)$		$\nabla^2 f(a)$	
		$\nabla f(a+h)$		$\nabla^3 f(a)$
$a+2h$	$f(a+2h)$		$\nabla^2 f(a+h)$	
		$\nabla f(a+2h)$		$\nabla^3 f(a+h)$
$a+3h$	$f(a+3h)$		$\nabla^2 f(a+2h)$	
		$\nabla f(a+3h)$		
$a+4h$	$f(a+4h)$			

1.7 The Difference Operator ∇ :

The symbol ∇ is used for backward difference operation. It is an operator and this operator is called "Nabla". $\nabla^2 f(x)$ stands for second time backward operations (two times operations) for the function $f(x)$. Similarly, $\nabla^n f(x)$ stands for n times backward operations for the function $f(x)$.

1.8 Worked out Examples :

Example (1) : Construct a backward difference table from the following values of x and y :

x :	5	10	15	20	25
y :	425	523	618	705	789

Solution : Here, $y = f(x)$.

$\therefore$ Backward difference Table is constructed as follows :

Argument x	Entry $f(x)$	1st Difference $\nabla f(x)$	2nd Difference $\nabla^2 f(x)$	3rd Difference $\nabla^3 f(x)$	4th Difference $\nabla^4 f(x)$
5	425				
		98			
10	523		-3		
		95		-5	
15	618		-8		10
		87		5	
20	705		-3		
		84			
25	789				

Example (2) : Construct a backward difference table with the following data :

$$u_0 = 3, \quad u_1 = 14, \quad u_2 = 87, \quad u_3 = 223, \quad u_4 = 529$$

Solution : Here, $y = u_x = f(x)$.

$\therefore$ Backward difference Table is constructed as follows :

Argument x	Entry u_x	1st Difference ∇u_x	2nd Difference $\nabla^2 u_x$	3rd Difference $\nabla^3 u_x$	4th Difference $\nabla^4 u_x$
0	3				
		11			
1	14		62		
		73		1	
2	87		63		106
		136		107	
3	223		170		
		306			
4	529				