General Solution of Homogeneous Equation of Second Order

Equation with right hand member is zero

Case (i) : If the roots are distinct : Suppose m_1, m_2 are all distinct roots, then the general solution is

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ where C_1, C_2 are arbitrary constants.

Case (ii) : If the roots are equal : Suppose $m_1 = m_2 = m$ (say), then the general solution is

$y = (C_1 + C_2 x)e^{mx}$ where C_1, C_2 are arbitrary constants.

Case (iii) : If the roots are complex numbers : Suppose $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ having a pair of complex roots, then the general solution is

$y = e^{\alpha x}[A\ cos\ \beta x + B \sin \beta x]$ where A, B are arbitrary constants.

Exercise

1. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$

$Solution : Given\ equation\ is,$

$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$ …………………… (i)

Suppose, $y = e^{mx}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore m^2 e^{mx} + 5me^{mx} + 4e^{mx} = 0$

$\Rightarrow (m^2 + 5m + 4)e^{mx} = 0$

$\Rightarrow m^2 + 5m + 4 = 0\ \ [\because\ e^{mx} \neq 0]$

$\Rightarrow (m + 4)(m + 1) = 0$

$\Rightarrow m = -4, -1$

$\therefore$ *The general solution is* $y = C_1e^{-4x} + C_2e^{-x}$ **Answer**

2. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Solution : Given equation is,

$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$……………………..(i)

Suppose, $y = e^{mx}$ *is the trial solution of the given equation.*

$\therefore m^2e^{mx} - 7me^{mx} + 12e^{mx} = 0$

$\Rightarrow (m^2 - 7m + 12)e^{mx} = 0$

$\Rightarrow m^2 - 7m + 12 = 0 \;\; [\because \; e^{mx} \neq 0]$

$\Rightarrow (m-4)(m-3) = 0$

$\Rightarrow m = 4, 3$

$\therefore$ *The general solution is* $y = C_1e^{4x} + C_2e^{3x}$ **Answer**

3. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Solution : Given equation is,

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$……………………..(i)

Suppose, $y = e^{mx}$ *is the trial solution of the given equation.*

$\therefore m^2e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$\Rightarrow (m^2 - 3m + 2)e^{mx} = 0$

$\Rightarrow m^2 - 3m + 2 = 0 \;\; [\because \; e^{mx} \neq 0]$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 2, 1$

$\therefore \textit{The general solution is } y = C_1 e^{2x} + C_2 e^{x}$ **Answer**

4. $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

Solution : Given equation is,

$\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$……………………(i)

Suppose, $y = e^{mx}$ *is the trial solution of the given equation.*

$\therefore m^2 e^{mx} + (a+b)me^{mx} + abe^{mx} = 0$

$\Rightarrow \{m^2 + (a+b)m + ab\}e^{mx} = 0$

$\Rightarrow m^2 + (a+b)m + ab = 0 \ [\because \ e^{mx} \neq 0]$

$\Rightarrow (m+a)(m+b) = 0$

$\Rightarrow m = -a, -b$

$\therefore \textit{The general solution is } y = C_1 e^{-ax} + C_2 e^{-bx}$ **Answer**

5.(i) $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

Solution : Given equation is,

$2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$……………………….(i)

Suppose, $y = e^{mx}$ *is the trial solution of the given equation.*

$\therefore 2m^2 e^{mx} - 3me^{mx} + e^{mx} = 0$

$\Rightarrow (2m^2 - 3m + 1)e^{mx} = 0$

$\Rightarrow 2m^2 - 3m + 1 = 0 \ [\because \ e^{mx} \neq 0]$

$\Rightarrow (2m-1)(m-1)=0$

$\Rightarrow m=\frac{1}{2},1$

$\therefore \textit{The general solution is } y=C_1e^{\frac{x}{2}}+C_2e^{x}$ **Answer**

(ii) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

Solution : Given equation is,

$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$.........................(i)

Suppose, $y=e^{mx}$ *is the trial solution of the given equation.*

$\therefore m^2e^{mx}+2me^{mx}+e^{mx}=0$

$\Rightarrow (m^2+2m+1)e^{mx}=0$

$\Rightarrow m^2+2m+1=0\ [\because\ e^{mx}\neq 0]$

$\Rightarrow (m+1)^2=0$

$\Rightarrow m=-1.-1$

$\therefore \textit{The general solution is } y=(C_1+C_2x)e^{-x}$ **Answer**

8. $(D^2-2mD+m^2+n^2)y=0$

Solution : Given equation is,

$(D^2-2mD+m^2+n^2)y=0$

$\Rightarrow D^2y-2mDy+(m^2+n^2)y=0$ ....(i)

Suppose, $y=e^{kx}$ *is the trial solution of the given equation.*

$\therefore k^2e^{kx}-2mke^{kx}+(m^2+n^2)e^{kx}=0$

$\Rightarrow \{k^2 - 2mk + (m^2 + n^2)\}e^{kx} = 0$

$\Rightarrow k^2 - 2mk + (m^2 + n^2) = 0 \;\; [\because \; e^{kx} \neq 0]$

$\Rightarrow k = \frac{-(-2m) \pm \sqrt{(-2m)^2 - 4.1.(m^2+n^2)}}{2.1}$

$\Rightarrow k = \frac{2m \pm \sqrt{4m^2 - 4m^2 - 4n^2}}{2}$

$\Rightarrow k = \frac{2m \pm \sqrt{-4n^2}}{2}$

$\Rightarrow k = \frac{2m \pm 2in}{2}$

$\Rightarrow k = m \pm in$

$\Rightarrow k = m + in, m - in$

$\therefore The\ general\ solution\ is\ y = C_1 e^{(m+in)x} + C_2 e^{(m-in)x}$

Or, $The\ general\ solution\ is\ y = e^{mx}(Acosnx + Bsinnx)$ **Answer**

9. (i) $(D^2 - 4D + 13)y = 0$

$Solution : Given\ equation\ is,$

$(D^2 - 4D + 13)y = 0$

$\Rightarrow D^2y - 4Dy + 13y = 0$ ……………….(i)

Suppose, $y = e^{kx}\ is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore k^2e^{kx} - 4ke^{kx} + 13e^{kx} = 0$

$\Rightarrow (k^2 - 4k + 13)e^{kx} = 0$

$\Rightarrow k^2 - 4k + 13 = 0 \;\; [\because \; e^{kx} \neq 0]$

$\Rightarrow k = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.13}}{2.1}$

$\Rightarrow k = \frac{4 \pm \sqrt{16-52}}{2}$

$\Rightarrow k = \frac{4 \pm \sqrt{-36}}{2}$

$\Rightarrow k = \frac{4 \pm 6i}{2}$

$\Rightarrow k = 2 \pm 3i$

$\Rightarrow k = 2 + 3i, 2 - 3i$

$\therefore The\ general\ solution\ is\ y = C_1 e^{(2+3i)x} + C_2 e^{(2-3i)x}$

Or, $The\ general\ solution\ is\ y = e^{2x}(Acos3x + Bsin3x)$ **Answer**

(ii) $(D^2 - n^2)y = 0$

$Solution : Given\ equation\ is,$

$(D^2 - n^2)y = 0$

$\Rightarrow D^2 y - n^2 y = 0$ ………………..(i)

Suppose, $y = e^{kx}\ is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore k^2 e^{kx} - n^2 e^{kx} = 0$

$\Rightarrow (k^2 - n^2)e^{kx} = 0$

$\Rightarrow k^2 - n^2 = 0\ \ [\because\ e^{kx} \neq 0]$

$\Rightarrow k = \pm n$

$\Rightarrow k = n, -n$

$\therefore The\ general\ solution\ is\ y = C_1 e^{nx} + C_2 e^{-nx}$ **Answer**

10. (i) $\frac{d^2s}{dt^2} + 4\frac{ds}{dt} + 13s = 0$

$Solution : Given\ equation\ is,$

$\frac{d^2s}{dt^2} + 4\frac{ds}{dt} + 13s = 0$……………………..(i)

Suppose, $s = e^{kt}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore k^2e^{kt} + 4ke^{kt} + 13e^{kt} = 0$

$\Rightarrow (k^2 + 4k + 13)e^{kt} = 0$

$\Rightarrow k^2 + 4k + 13 = 0\ \ [\because\ e^{kt} \neq 0]$

$\Rightarrow k = \frac{-4 \pm \sqrt{4^2-4.1.13}}{2.1}$

$\Rightarrow k = \frac{-4 \pm \sqrt{16-52}}{2}$

$\Rightarrow k = \frac{-4 \pm \sqrt{-36}}{2}$

$\Rightarrow k = \frac{-4 \pm 6i}{2}$

$\Rightarrow k = -2 \pm 3i$

$\therefore The\ general\ solution\ is,\qquad s = C_1e^{(-2+3i)t} + C_2e^{(-2-3i)t}$

Or, $The\ general\ solution\ is,\ \ s = e^{-2t}(Acos3t + Bsin3t)$ **Answer**

(ii) $(D + 3)^2y = 0$

$Solution : Given\ equation\ is,$

$(D + 3)^2y = 0$

$\Rightarrow D^2y + 6Dy + 9y = 0$ ………………..(i)

Suppose, $y = e^{kx}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore k^2e^{kx} + 6ke^{kx} + 9e^{kx} = 0$

$\Rightarrow (k^2 + 6k + 9)e^{kx} = 0$

$\Rightarrow k^2 + 6k + 9 = 0 \;\; [\because \; e^{kx} \neq 0]$

$\Rightarrow (k+3)^2 = 0$

$\Rightarrow k = -3, -3$

$\therefore \textit{The general solution is } y = (C_1 + C_2x)e^{-3x}$ **Answer**

11. Solve in the particular cases :

(i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$; when $x = 0, y = 3$ and $\frac{dy}{dx}$ =0

$Solution : Given\ equation\ is,$

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ ………………. (i)

Suppose, $y = e^{mx}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore m^2e^{mx} + me^{mx} - 2e^{mx} = 0$

$\Rightarrow (m^2 + m - 2)e^{mx} = 0$

$\Rightarrow m^2 + m - 2 = 0 \;\; [\because \; e^{mx} \neq 0]$

$\Rightarrow (m-1)(m+2) = 0$

$\Rightarrow m = 1. -2$

$\therefore \textit{The general solution is } y = C_1e^x + C_2e^{-2x}$ …………… (ii)

(ii) $\Rightarrow \frac{dy}{dx} = C_1e^x - 2C_2e^{-2x}$ ……………………….. (iii)

$When\ x = 0, y = 3\ then\ from\ (ii), we\ get$

$3 = C_1 + C_2$ .... (iv)

$When\ x = 0, \dfrac{dy}{dx} = 0\ then\ from\ (iii), we\ get$

$0 = C_1 - 2C_2$ .... (v)

From, (iv) & (v), we get $\quad C_1 = 2 \ \ \& \ \ C_2 = 1$

$\therefore$ (ii) $\Rightarrow \quad y = 2e^x + e^{-2x}$ **Answer**

(ii) $\frac{d^2y}{dx^2} + y = 0$; when $x = 0, y = 4$ and $x = \frac{\pi}{2},\ y = 0$

$Solution : Given\ equation\ is,$

$\frac{d^2y}{dx^2} + y = 0$ .... (i)

Suppose, $y = e^{mx}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore m^2e^{mx} + e^{mx} = 0$

$\Rightarrow (m^2 + 1)e^{mx} = 0$

$\Rightarrow m^2 + 1 = 0 \quad [\because\ e^{mx} \neq 0]$

$\Rightarrow m = 0 \pm i$

$The\ general\ solution\ is,\ y = A\cos x + B\sin x$ .... (ii)

When $x = 0, y = 4$ then from (ii), we get

$4 = A\cos 0 + B\sin 0$

$\Rightarrow 4 = A$

When $x = \frac{\pi}{2}, y = 0$ then from (ii), we get

$0 = A\cos\dfrac{\pi}{2} + B\sin\dfrac{\pi}{2}$

$\Rightarrow 0 = \mathrm{B}$

$Putting\ A = 4\ and\ B = 0\ in\ (ii), we\ get$

$y = 4\cos x + 0. sinx$

$\Rightarrow y = 4\cos x$ **Answer.**

(iii) $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$; when $t = 0, x = 0$ and $\frac{dx}{dt} = 0$

$Solution : Given\ equation\ is,$

$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$ .... (i)

Suppose, $x = e^{mt}$ $is\ the\ trial\ solution\ of\ the\ given\ equation.$

$\therefore m^2e^{mt} - 3me^{mt} + 2e^{mt} = 0$

$\Rightarrow (m^2 - 3m + 2)e^{mt} = 0$

$\Rightarrow m^2 - 3m + 2 = 0\ \ [\because\ e^{mt} \neq 0]$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

$\therefore The\ general\ solution\ is\ x = C_1e^t + C_2e^{2t}$ .... (ii)

When $t = 0\ and\ x = 0$, then from (ii), we get

$0 = C_1 + C_2$ .... (iii)

(ii) $\Rightarrow \frac{dx}{dt} = C_1e^t + 2C_2e^{2t}$ .... (iv)

When $t = 0\ and\ \frac{dx}{dt} = 0$, then from (iv), we get

$0 = C_1 + 2C_2$ ....(v)

Solving (iii) & (v), we get

$C_1 = 0, C_2 = 0$

Putting values of C_1& C_2 in (ii), we get

$x = 0$ **Answer**