

Derivation of PDE by Rule II

Ex 1. Form PDE from $\varphi(x + y + z, x^2 + y^2 - z^2) = 0$ by eliminating φ , where φ is an arbitrary function. What is the order of this PDE ?

Solution : Given, $\varphi(x + y + z, x^2 + y^2 - z^2) = 0$ (i)

Let us consider, $u = x + y + z$ and $v = x^2 + y^2 - z^2$ (ii)

Then, (i) reduces to $\varphi(u, v) = 0$ (iii)

Differentiating (iii) partially w.r.t.x, we get

$$\frac{\partial \varphi}{\partial u} \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right) + \frac{\partial \varphi}{\partial v} \left(\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0 \quad \dots \dots \dots \text{(iv)}$$

Differentiating (iii) partially w.r.t.y, we get

$$\frac{\partial \varphi}{\partial u} \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) + \frac{\partial \varphi}{\partial v} \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z} \right) = 0 \quad \dots \dots \dots \text{(v)}$$

From (ii), we get,

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial z} = 1, \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial z} = -2z, \quad \frac{\partial u}{\partial y} = 1, \frac{\partial v}{\partial y} = 2y \quad \dots \dots \dots \text{(vi)}$$

From (iv) and (vi), we get,

$$\frac{\partial \varphi}{\partial u} (1 + p \cdot 1) + \frac{\partial \varphi}{\partial v} \{2x + p \cdot (-2z)\} = 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial u} (1 + p) = -\frac{\partial \varphi}{\partial v} \{2(x - pz)\}$$

$$\Rightarrow \frac{\frac{\partial \varphi}{\partial u}}{\frac{\partial \varphi}{\partial v}} = \frac{2(pz-x)}{1+p} \quad \dots \dots \dots \text{(vii)}$$

From (v) and (vi), we get,

$$\frac{\partial \varphi}{\partial u} (1 + q \cdot 1) + \frac{\partial \varphi}{\partial v} \{2y + q \cdot (-2z)\} = 0$$

$$\Rightarrow \frac{\partial \varphi}{\partial u} (1 + q) = -\frac{\partial \varphi}{\partial v} \{2(y - qz)\}$$

$$\Rightarrow \frac{\frac{\partial \varphi}{\partial u}}{\frac{\partial \varphi}{\partial v}} = \frac{2(qz-y)}{1+q} \dots \dots \dots \text{(viii)}$$

Dividing (vii) by (viii), we get

$$1 = \frac{\frac{2(pz-x)}{1+p}}{\frac{2(qz-y)}{1+q}}$$

$$\Rightarrow \frac{2(qz-y)}{1+q} = \frac{2(pz-x)}{1+p}$$

$$\Rightarrow (1+p)(qz-y) = (1+q)(pz-x)$$

$$\Rightarrow qz + pqz - y - py = pz + pqz - x - qx$$

$$\Rightarrow (y+z)p - (z+x)q = x - y, \text{ which is the required PDE.}$$

This is the 1st order PDE.

Exercise 1B

1. $z = e^{mx}\varphi(x+y)$

Solution : Given, $z = e^{mx}\varphi(x+y) \dots \dots \dots \text{(i)}$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = me^{mx}\varphi(x+y) + e^{mx}\varphi'(x+y).1$$

$$\Rightarrow p = mz + e^{mx}\varphi'(x+y) \quad [\text{From (i)}]$$

$$\Rightarrow p - mz = e^{mx}\varphi'(x+y) \dots \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = e^{mx}\varphi'(x+y).1$$

$$\Rightarrow q = e^{mx}\varphi'(x+y) \dots \dots \dots \text{(iii)}$$

Dividing (ii) by (iii) with their corresponding sides, we get,

$$\frac{p - mz}{q} = 1$$

$$\Rightarrow p - mz = q$$

$$\Rightarrow p - q = mz, \text{ which is the required PDE.}$$

2. $z = f(x + ay)$

Solution : Given, $z = f(x + ay)$(i)

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = f'(x + ay). 1$$

$$\Rightarrow p = f'(x + ay) \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = f'(x + ay). a$$

$$\Rightarrow q = a.f'(x + ay) \text{(iii)}$$

Dividing (iii) by (ii), we get

$$\frac{q}{p} = a$$

$$\Rightarrow q = ap, \text{ which s the required PDE.}$$

3. $z = xy + f(x^2 + y^2)$

Solution : Given, $z = xy + f(x^2 + y^2)$ (i)

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = y + f'(x^2 + y^2). 2x$$

$$\Rightarrow p - y = 2x.f'(x^2 + y^2) \text{ (ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = x + f'(x^2 + y^2) \cdot 2y$$

$$\Rightarrow q - x = 2y \cdot f'(x^2 + y^2) \dots \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{p - y}{q - x} = \frac{x}{y}$$

$$\Rightarrow py - y^2 = qx - x^2$$

$\Rightarrow py - qx = y^2 - x^2$, which s the required PDE.

4. $z = x + y + f(xy)$

Solution : Given, $z = x + y + f(xy) \dots \text{(i)}$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = 1 + f'(xy) \cdot y$$

$$\Rightarrow p - 1 = y \cdot f'(xy) \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = 1 + f'(xy) \cdot x$$

$$\Rightarrow q - 1 = x \cdot f'(xy) \dots \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{p - 1}{q - 1} = \frac{y}{x}$$

$$\Rightarrow px - x = qy - y$$

$\Rightarrow px - qy = x - y$, which s the required PDE.

$$5. z = f\left(\frac{xy}{z}\right)$$

Solution : Given, $z = f\left(\frac{xy}{z}\right) \dots \dots \dots \text{(i)}$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = f' \left(\frac{xy}{z} \right) \cdot \frac{y}{z}$$

$$\Rightarrow p = f' \left(\frac{xy}{z} \right) \cdot \frac{y}{z} \dots \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = f' \left(\frac{xy}{z} \right) \cdot \frac{x}{z}$$

$$\Rightarrow q = f' \left(\frac{xy}{z} \right) \cdot \frac{x}{z} \dots \dots \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{p}{q} = \frac{y}{x}$$

$$\Rightarrow px = qy$$

$\Rightarrow px - qy = 0$, which is the required PDE.

$$6. z = f(x - y)$$

Solution : Given, $z = f(x - y) \dots \dots \dots \text{(i)}$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = f'(x - y) \cdot 1$$

$$\Rightarrow p = f'(x - y) \dots \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = f'(x-y) \cdot (-1)$$

$$\Rightarrow q = (-1) \cdot f'(x-y) \dots \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{p}{q} = -1$$

$$\Rightarrow p = -q$$

$\Rightarrow p + q = 0$, which is the required PDE

$$7. z = (x-y)\varphi(x^2+y^2)$$

$$\text{Solution : Given, } z = (x-y)\varphi(x^2+y^2) \dots \text{(i)}$$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = 1 \cdot \varphi(x^2+y^2) + (x-y)\varphi'(x^2+y^2) \cdot 2x$$

$$\Rightarrow p = \varphi(x^2+y^2) + (x-y)\varphi'(x^2+y^2) \cdot 2x$$

$$\Rightarrow (x-y)p = (x-y)\varphi(x^2+y^2) + (x-y)^2\varphi'(x^2+y^2) \cdot 2x$$

$$\Rightarrow (x-y)p = z + (x-y)^2\varphi'(x^2+y^2) \cdot 2x$$

$$\Rightarrow (x-y)p - z = 2x \cdot (x-y)^2\varphi'(x^2+y^2) \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = (-1) \cdot \varphi(x^2+y^2) + (x-y)\varphi'(x^2+y^2) \cdot 2y$$

$$\Rightarrow q = (-1) \cdot \varphi(x^2+y^2) + (x-y)\varphi'(x^2+y^2) \cdot 2y$$

$$\Rightarrow (x-y)q = (-1) \cdot (x-y)\varphi(x^2+y^2) + (x-y)^2\varphi'(x^2+y^2) \cdot 2y$$

$$\Rightarrow (x-y)q = -z + (x-y)^2\varphi'(x^2+y^2) \cdot 2y$$

$$\Rightarrow (x-y)q + z = 2y \cdot (x-y)^2\varphi'(x^2+y^2) \dots \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{(x-y)p-z}{(x-y)q+z} = \frac{x}{y}$$

$$\Rightarrow (x-y)yp - yz = (x-y)xq + zx$$

$\Rightarrow (x-y)yp - (x-y)xq = yz + zx$, which is the required PDE.

8. $z = f(x^2 + 2y^2)$

Solution : Given, $z = f(x^2 + 2y^2)$(i)

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = f'(x^2 + 2y^2). 2x$$

$$\Rightarrow p = 2x.f'(x^2 + 2y^2) \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = f'(x^2 + 2y^2). 4y$$

$$\Rightarrow q = 4y.f'(x^2 + 2y^2) \text{(iii)}$$

Dividing (ii) by (iii), we get

$$\frac{p}{q} = \frac{x}{2y}$$

$$\Rightarrow 2py = qx$$

$\Rightarrow 2py - qx = 0$, which is the required PDE.

9. $x = f(z) + g(y)$

Solution : Given, $x = f(z) + g(y)$(i)

Differentiating (i) partially w.r.t.x, we get,

$$1 = f'(z) \cdot \frac{\partial z}{\partial x}$$

$$\Rightarrow 1 = f'(z) \cdot \frac{\partial z}{\partial x} \dots \text{(ii)}$$

Differentiating (i) partially w.r.t.y, we get,

$$0 = f'(z) \cdot \frac{\partial z}{\partial y} + g'(y) \dots \text{(iii)}$$

Differentiating (ii) partially w.r.t.x, we get,

$$0 = f''(z) \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial x} + f'(z) \cdot \frac{\partial^2 z}{\partial x^2}$$

$$\Rightarrow 0 = f''(z) \cdot p \cdot p + f'(z) \cdot r$$

$$\Rightarrow -f'(z) \cdot r = f''(z) \cdot p \cdot p \dots \text{(iv)}$$

Differentiating (iii) partially w.r.t.x, we get,

$$0 = f''(z) \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + f'(z) \cdot \frac{\partial^2 z}{\partial x \partial y}$$

$$\Rightarrow 0 = f''(z) \cdot p \cdot q + f'(z) \cdot s$$

$$\Rightarrow -f'(z) \cdot s = f''(z) \cdot p \cdot q \dots \text{(v)}$$

Dividing (iv) by (v), we get Type equation here.

$$\frac{r}{s} = \frac{p}{q}$$

$$\Rightarrow qr = ps$$

$\Rightarrow ps - qr = 0$, which is the required PDE.

10. $\mathbf{z} = \mathbf{f}(y + ax) + \mathbf{g}(y + bx)$, $a \neq b$

Solution : Given, $z = f(y + ax) + g(y + bx) \dots \text{(i)}$

Differentiating (i) partially w.r.t.x, we get,

$$\frac{\partial z}{\partial x} = f'(y + ax) \cdot a + g'(y + bx) \cdot b \dots \dots \dots \text{(ii)}$$

Again, differentiating (ii) partially w.r.t.x, we get,

$$\frac{\partial^2 z}{\partial x^2} = f''(y + ax) \cdot a^2 + g''(y + bx) \cdot b^2$$

$$\Rightarrow r = a^2 \cdot f''(y + ax) + b^2 \cdot g''(y + bx) \dots \dots \dots \text{(iii)}$$

Differentiating (ii) partially w.r.t.y, we get,

$$\frac{\partial^2 z}{\partial y \partial x} = a \cdot f''(y + ax) + b \cdot g''(y + bx)$$

$$\Rightarrow s = a \cdot f''(y + ax) + b \cdot g''(y + bx) \dots \dots \dots \text{(iv)}$$

Differentiating (i) partially w.r.t.y, we get,

$$\frac{\partial z}{\partial y} = f'(y + ax) + g'(y + bx) \dots \dots \dots \text{(v)}$$

Again, differentiating (v) partially w.r.t.y, we get,

$$\frac{\partial^2 z}{\partial y^2} = f''(y + ax) + g''(y + bx)$$

$$\Rightarrow t = f''(y + ax) + g''(y + bx) \dots \dots \dots \text{(vi)}$$

From (iii), (iv) and (vi), we get

$$\begin{aligned} r - (a + b)s + abt &= a^2 \cdot f''(y + ax) + b^2 \cdot g''(y + bx) \\ &\quad - (a + b)\{a \cdot f''(y + ax) + b \cdot g''(y + bx)\} \\ &\quad + ab\{f''(y + ax) + g''(y + bx)\} \\ &= a^2 \cdot f''(y + ax) + b^2 \cdot g''(y + bx) - a^2 \cdot f''(y + ax) \\ &\quad - ab \cdot f''(y + ax) - ab \cdot g''(y + bx) - b^2 \cdot g''(y + bx) \\ &\quad + ab \cdot f''(y + ax) + ab \cdot g''(y + bx) \end{aligned}$$

$\Rightarrow r - (a + b)s + abt = 0$, which is the required PDE.